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T-Duality and Two-Loop Renormalization Flows*

Peter E. Haagensen[1] and Kasper Olsen[1,2]

[1] *Center for Theoretical Physics*
Laboratory for Nuclear Science
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139, USA

and

[2] *The Niels Bohr Institute*
Blegdamsvej 17
DK 2100 Copenhagen
DENMARK

Abstract

Manifest T-duality covariance of the one-loop renormalization group flows is shown for a generic bosonic sigma model with an abelian isometry, by referring a set of previously derived consistency conditions to the tangent space of the target. For a restricted background, T-duality transformations are then studied at the next order, and the ensuing consistency conditions are found to be satisfied by the two-loop Weyl anomaly coefficients of the model. This represents an extremely non-trivial test of the covariance of renormalization group flows under T-duality, and a stronger condition than T-duality invariance of the string background effective action.

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E-mail: `haagense@ctp.mit.edu, olsen@ctp.mit.edu`.

1. Introduction

When one thinks about symmetries in quantum field theory, the examples that are likely to come to mind are typically of transformations which act on the fields of a theory: for instance, gauge or flavor symmetries, or yet charge conjugation, parity, and time reversal. Less commonly, some theories might also possess symmetries that act on their parameter space. The prototype of such a symmetry occurs actually not in field theory but in lattice spin or gauge systems, where it is known as Kramers-Wannier duality (or a generalization thereof), and it states that some system at low temperature is equivalent – or dual – to some other system at high temperature.

One reason such symmetries are interesting is the fact that they act on the same (parameter) space in which also acts another important and ubiquitous symmetry: the renormalization group (RG). Similarly to the Kramers-Wannier dualities mentioned above, the RG represents transformations in the parameter space of a theory that leave a partition function invariant. Seen in this light, then, it becomes natural to investigate the interplay between the RG and other symmetries acting on the parameter space of a theory.

In the context of quantum field theory one such symmetry is target space duality (T-duality, for short) [1], present in $d = 2$ nonlinear sigma models with a target abelian isometry (we will furthermore restrict ourselves to bosonic models on closed worldsheets). In its simplest incarnation, it relates the partition function for a bosonic string compactified on a torus of radius R to the same partition function evaluated at radius α'/R. In general, however, in order to identify a T-duality symmetry (at least to lowest nontrivial order in α', in which case we refer to it as *classical* duality symmetry), all that is required is that the target possess an abelian isometry, so that actually a large class of targets enjoys such a symmetry.

In spite of its being inevitably tied to the language of string theory, it is clear that T-duality symmetry presents an interest quite independently of its relevance for string theory. It is with this interest in mind that we approach the issue of two-loop corrections to duality transformations.

The easiest path to T-duality transformations is probably the one given by Buscher [2], in which the path integral is considered for a generic sigma model with an abelian target isometry. The isometry is gauged, and this leads to two possible background descriptions of the same path integral, obtained by performing the trivial parts of the path integral in a different order. This derivation, valid at first nontrivial order ($\mathcal{O}(\alpha')$), is classic by now, and we will not reproduce it. The fields which do not undergo this gauging procedure simply play a spectator role and, apart from a Jacobian factor which appears in the course of performing these integrations (leading to a dilaton shift in the transformations), the procedure is simple enough that one may imagine there should not be any significant

difference at higher orders. There emerges the vague expectation that while one might obtain more complicated Jacobian factors, leading to more complicated dilaton shifts, classical duality transformations should otherwise remain just as good a symmetry.

Unfortunately this expectation does not seem to be realized. In a related development, Tseytlin found in [3] that for a particular set of backgrounds, classical duality transformations did *not* keep the two-loop string background effective action invariant, and showed what the needed corrections to the transformations were so that the effective action would indeed be invariant at that order. Even in that restrictive case the necessary two-loop corrections did not amount to any field redefinition arising from a target reparametrization. Other authors have also noticed a similar breakdown of classical duality symmetry [4].

Independently of the existence at all of a string background effective action, but rather from a purely 2d field theoretical point of view, it was found in [5] that the requirement of consistency between the sigma model RG flow and classical duality transformations imposed stringent conditions on the beta functions of the model. For a generic background, these conditions are satisfied by, and only by, the correct and well-known one-loop beta functions. It is this consistency which concerns us here, and we are led to the main question we shall address: what do possible two-loop corrections to duality transformations entail for the consistency between T-duality symmetry and the RG?

In Section 2, we phrase this consistency requirement as the commutation of two motions in the space of the theory, defined through the action of T-duality and through the Weyl anomaly coefficients, respectively. We then show that the consistency conditions presented in [5], which appear complicated and unwieldy, can be stated in a compact expression, by referring the Weyl anomaly coefficients to the tangent space of the target. This expresses the manifest covariance of the RG flow under the duality group $\mathbb{Z}_2$. We then examine how this may be modified at higher order.

While classical duality transformations may not be a symmetry of the two-loop string effective action, the indication so far is that local perturbative modifications of these transformations do exist such that a T-duality symmetry can still be defined at higher orders in α'. In Section 3, we reestablish contact with the string background effective action, considering modified duality transformations at higher order, and we analyze to what extent the consistency conditions may or may not imply duality invariance of the background effective action and vice-versa. If the duality transformations themselves are modified at next order, that will induce modifications in the consistency conditions, and *a priori* there is no telling what may happen with the relation between the two-loop RG flow and the two-loop duality transformations. Some authors [6] have suggested that the simple connection between beta functions and duality transformations found at order α' breaks down, and that in fact T-duality at two loops does not even map sigma models into sigma models. Given the particularly neat and compact expression of RG covariance under one-loop duality shown

in Section 2, it would be quite disappointing to see this entire structure simply dismantled at the next order. In Section 4 we consider a particular class of backgrounds and show that, quite on the contrary, the consistency between duality and the RG remains alive and well at two-loop order, in that the two-loop beta functions perfectly fit the consistency conditions engendered by duality at two loops, and in particular fixed points of one theory are mapped to fixed points of its dual.

In Section 5 we present some conclusions and outlook, and in the Appendix we collect the formulas summarizing the Kaluza-Klein reduction relevant to the verification of the consistency conditions.

2. Manifest Covariance at One-Loop Order

We consider a $d{=}2$ bosonic sigma model on a generic $(D{+}1)$-dimensional background of metric $g_{\mu\nu}(X)$ and antisymmetric tensor $b_{\mu\nu}(X)$, with the requirement that it have an abelian isometry in one target direction. Furthermore, we choose (adapted) target coordinates $X^\mu = (\theta, X^i)$ such that the isometry lies in the θ-direction. In these coordinates, all background fields depend only on X^i. The sigma model action is

$$\begin{aligned} S = \frac{1}{4\pi\alpha'} \int d^2\sigma \; \big[& g_{00}(X)\partial_\alpha\theta\partial^\alpha\theta + 2g_{0i}(X)\partial_\alpha\theta\partial^\alpha X^i + g_{ij}(X)\partial_\alpha X^i\partial^\alpha X^j + \\ & i\varepsilon^{\alpha\beta}\left(2b_{0i}(X)\partial_\alpha\theta\partial_\beta X^i + b_{ij}(X)\partial_\alpha X^i\partial_\beta X^j\right)\big] \; . \end{aligned} \tag{2.1}$$

With proper regularization and renormalization in place, renormalized background couplings become functions of a subtraction scale μ, so that they encode the RG flow of the model through their dependence on μ. Classical duality transformations are given by:

$$\begin{aligned} \tilde{g}_{00} &= \frac{1}{g_{00}} \; , \\ \tilde{g}_{0i} &= \frac{b_{0i}}{g_{00}} \; , \quad \tilde{b}_{0i} = \frac{g_{0i}}{g_{00}} \; , \\ \tilde{g}_{ij} &= g_{ij} - \frac{g_{0i}g_{0j} - b_{0i}b_{0j}}{g_{00}} \; , \\ \tilde{b}_{ij} &= b_{ij} - \frac{g_{0i}b_{0j} - b_{0i}g_{0j}}{g_{00}} \; , \end{aligned} \tag{2.2}$$

mapping the background $\{g_{\mu\nu}, b_{\mu\nu}\}$ onto a dual background $\{\tilde{g}_{\mu\nu}, \tilde{b}_{\mu\nu}\}$. On a curved worldsheet another background coupling is required to ensure renormalizability, that of the dilaton $\phi(X)$ to the worldsheet scalar curvature. In our approach [5], we initially leave the dilaton transformation under duality unspecified until consistency conditions are enforced, at which point it becomes uniquely determined to be

$$\tilde{\phi} = \phi - \frac{1}{2}\ln g_{00} \; . \tag{2.3}$$

The RG flow of the background couplings is given by their respective beta functions:

$$\beta^g_{\mu\nu} \equiv \mu \frac{d}{d\mu} g_{\mu\nu} \ , \quad \beta^b_{\mu\nu} \equiv \mu \frac{d}{d\mu} b_{\mu\nu} \ , \quad \beta^\phi \equiv \mu \frac{d}{d\mu} \phi \ , \tag{2.4}$$

while the unintegrated (worldsheet) stress-energy trace, which yields the conformal anomaly of the model, is determined from the Weyl anomaly coefficients, given by [7]:

$$\begin{aligned} \bar{\beta}^g_{\mu\nu} &= \beta^g_{\mu\nu} + 2\alpha' \nabla_\mu \partial_\nu \phi \\ \bar{\beta}^b_{\mu\nu} &= \beta^b_{\mu\nu} + \alpha' H_{\mu\nu}{}^\lambda \partial_\lambda \phi \\ \bar{\beta}^\phi &= \beta^\phi + \alpha' (\partial_\mu \phi)^2 \ . \end{aligned} \tag{2.5}$$

In previous work [5], the consistency conditions to be presented below were found to hold at lowest order both for the beta functions and the Weyl anomaly coefficients. However, while they are satisfied up to a target reparametrization by the beta functions (which is reasonable to expect), they are on the other hand *identically* satisfied by the Weyl anomaly coefficients (a deep reason for which, from the 2d field theory point of view, we have not found). Thus, although strictly speaking the Weyl anomaly coefficients do not represent an RG motion in the parameter space, in order to be concise, we will mainly be referring to these coefficients in what follows, making the distinction from the beta functions when necessary.

We now define an operation R, akin to the RG motion $\mu \frac{d}{d\mu}$:

$$R \begin{pmatrix} g_{\mu\nu} \\ b_{\mu\nu} \\ \phi \end{pmatrix} = \begin{pmatrix} \bar{\beta}^g_{\mu\nu}[g,b,\phi] \\ \bar{\beta}^b_{\mu\nu}[g,b,\phi] \\ \bar{\beta}^\phi[g,\phi] \end{pmatrix} \ , \tag{2.6}$$

so that on a generic functional $F[g,b,\phi]$,

$$RF[g,b,\phi] = \frac{\delta F}{\delta g_{\mu\nu}} \cdot \bar{\beta}^g_{\mu\nu} + \frac{\delta F}{\delta b_{\mu\nu}} \cdot \bar{\beta}^b_{\mu\nu} + \frac{\delta F}{\delta \phi} \cdot \bar{\beta}^\phi \ , \tag{2.7}$$

and a duality operation T:

$$T \begin{pmatrix} g_{\mu\nu} \\ b_{\mu\nu} \\ \phi \end{pmatrix} = \begin{pmatrix} \tilde{g}_{\mu\nu}[g,b] \\ \tilde{b}_{\mu\nu}[g,b] \\ \tilde{\phi}[g,\phi] \end{pmatrix} \ , \tag{2.8}$$

affecting the duality transformations (2.2), (2.3) and, more generally,

$$TF[g,b,\phi] = F[\tilde{g},\tilde{b},\tilde{\phi}] \ . \tag{2.9}$$

At any given order in α', R is defined by the corresponding Weyl anomaly coefficients, while *ab initio* we only know T at lowest order, where it is given by (2.2) and (2.3). Whether and how T is modified at higher orders is one of the crucial questions at hand.

Then, regardless of the order at which R is defined, *if* the action of T on $g_{\mu\nu}$ and $b_{\mu\nu}$ is defined by (2.2), the requirement that these two motions in the space of the theory commute:

$$[T, R] = 0 \ , \tag{2.10}$$

can be seen through (2.7) and (2.9) to be tantamount to the following consistency conditions [5]:

$$\begin{aligned}
\bar{\beta}^{\tilde{g}}_{00} &= -\frac{1}{g_{00}^2}\bar{\beta}^g_{00} \ , \\
\bar{\beta}^{\tilde{g}}_{0i} &= -\frac{1}{g_{00}^2}\left(b_{0i}\bar{\beta}^g_{00} - \bar{\beta}^b_{0i}g_{00}\right) \ , \\
\bar{\beta}^{\tilde{b}}_{0i} &= -\frac{1}{g_{00}^2}\left(g_{0i}\bar{\beta}^g_{00} - \bar{\beta}^g_{0i}g_{00}\right) \ , \\
\bar{\beta}^{\tilde{g}}_{ij} &= \bar{\beta}^g_{ij} - \frac{1}{g_{00}}\left(\bar{\beta}^g_{0i}g_{0j} + \bar{\beta}^g_{0j}g_{0i} - \bar{\beta}^b_{0i}b_{0j} - \bar{\beta}^b_{0j}b_{0i}\right) + \frac{1}{g_{00}^2}\left(g_{0i}g_{0j} - b_{0i}b_{0j}\right)\bar{\beta}^g_{00} \ , \\
\bar{\beta}^{\tilde{b}}_{ij} &= \bar{\beta}^b_{ij} - \frac{1}{g_{00}}\left(\bar{\beta}^g_{0i}b_{0j} + \bar{\beta}^b_{0j}g_{0i} - \bar{\beta}^g_{0j}b_{0i} - \bar{\beta}^b_{0i}g_{0j}\right) + \frac{1}{g_{00}^2}\left(g_{0i}b_{0j} - b_{0i}g_{0j}\right)\bar{\beta}^g_{00} \ .
\end{aligned} \tag{2.11}$$

As shown in [5], enforcing these conditions at $\mathcal{O}(\alpha')$ uniquely determines the beta functions at that order to be (up to a global factor):

$$\begin{aligned}
\beta^g_{\mu\nu} =& \alpha'\left(R_{\mu\nu} - \frac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho}\right) \ , \\
\beta^b_{\mu\nu} =& -\frac{\alpha'}{2}\nabla_\lambda H^\lambda{}_{\mu\nu} \ ,
\end{aligned} \tag{2.12}$$

where $H_{\mu\nu\lambda} = \partial_\mu b_{\nu\lambda} +$ cyclic permutations, and the dilaton transformation, or "shift", to be given by (2.3). Furthermore, one can now apply the same condition $[T, R] = 0$ to the dilaton transformation, obtaining yet another consistency condition:

$$\bar{\beta}^{\tilde{\phi}} = \bar{\beta}^\phi - \frac{1}{2}\frac{1}{g_{00}}\bar{\beta}^g_{00} \ . \tag{2.13}$$

This is satisfied for

$$\beta^\phi = C - \frac{\alpha'}{2}\nabla^2\phi \ , \tag{2.14}$$

with C an arbitrary constant, so that all beta functions are determined at one-loop order up to a global factor and the value of C .

For a model with n abelian isometries, one expects the duality symmetry to be $O(n,n,\mathbb{Z})$. (In the context of background effective actions, where the sigma model abelian isometry leads to spontaneous compactification, this was nicely shown in [8] and in [9]. The connection with the sigma model context is also explored in these and related works.) In our case, this is just $\mathbb{Z}_2$ ($O(1,1,\mathbb{R})$ is the group of hyperbolic rotations on the plane; restricting it to matrix representatives with integer entries eliminates all group elements but $\pm\mathbb{1}$). It is difficult to imagine that covariance under such a simple symmetry as $\mathbb{Z}_2$ cannot be expressed in terms simpler than (2.11). We now show that this is in fact possible if the tensors in (2.11) are referred to the tangent frame of the target.†

As in [5], we decompose the generic metric $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} = \begin{pmatrix} a & av_i \\ av_i & \bar{g}_{ij} + av_iv_j \end{pmatrix} , \tag{2.15}$$

so that $g_{00}=a$, $g_{0i}=av_i$, $g_{ij}=\bar{g}_{ij}+av_iv_j$. The components of the antisymmetric tensor are written as $b_{0i}\equiv w_i$ and b_{ij}. From (2.2) we find that in terms of this decomposition, the dual metric and antisymmetric tensor are given by the substitutions $a\to 1/a, v_i \leftrightarrow w_i$, and $\tilde{b}_{ij}=b_{ij}+w_iv_j-w_jv_i$. The vielbeins corresponding to (2.15) can always be taken in the block triangular form:

$$e_\mu{}^a = \begin{pmatrix} e_0{}^{\hat{0}} & e_0{}^\alpha \\ e_i{}^{\hat{0}} & e_i{}^\alpha \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 \\ \sqrt{a}v_i & \bar{e}_i{}^\alpha \end{pmatrix} , \tag{2.16}$$

where tangent space indices are decomposed as $a=\hat{0},\alpha$; $\alpha=1,2,\ldots,D$ (corresponding to the decomposition $\mu=0,i$; $i=1,2,\ldots,D$), and $\bar{e}_i{}^\alpha\bar{e}_j{}^\beta\delta_{\alpha\beta}=\bar{g}_{ij}$. For ease of reference, we also present here the inverse vielbein:

$$e_a{}^\mu = \begin{pmatrix} e_{\hat{0}}{}^0 & e_{\hat{0}}{}^i \\ e_\alpha{}^0 & e_\alpha{}^i \end{pmatrix} = \begin{pmatrix} 1/\sqrt{a} & 0 \\ -v_\alpha & \bar{e}_\alpha{}^i \end{pmatrix} , \tag{2.17}$$

with $v_\alpha \equiv \bar{e}_\alpha{}^i v_i$.

The tangent space Weyl anomaly coefficients are defined through:

$$\bar{\beta}^g_{ab} = e_a{}^\mu e_b{}^\nu \bar{\beta}^g_{\mu\nu} \quad , \quad \bar{\beta}^b_{ab} = e_a{}^\mu e_b{}^\nu \bar{\beta}^b_{\mu\nu} , \tag{2.18}$$

while an analogous definition holds in the dual background:

$$\bar{\beta}^{\tilde{g}}_{ab} = \tilde{e}_a{}^\mu \tilde{e}_b{}^\nu \bar{\beta}^{\tilde{g}}_{\mu\nu} \quad , \quad \bar{\beta}^{\tilde{b}}_{ab} = \tilde{e}_a{}^\mu \tilde{e}_b{}^\nu \bar{\beta}^{\tilde{b}}_{\mu\nu} . \tag{2.19}$$

† That the consistency conditions would simplify when referred to the tangent space was first noticed by P. Letourneau (private communication).

Using (2.11), it is now straightforward to work out the appropriate consistency relations for the tangent space anomaly coefficients. For illustration purposes, we show here the $\hat{0}\hat{0}$ and $\hat{0}\alpha$ components:

$$\bar{\beta}^{\tilde{g}}_{\hat{0}\hat{0}} = \tilde{e}_{\hat{0}}{}^{0}\ \tilde{e}_{\hat{0}}{}^{0}\ \bar{\beta}^{\tilde{g}}_{00} = a\ \bar{\beta}^{\tilde{g}}_{00} = -\frac{1}{a}\ \bar{\beta}^{g}_{00} = -e_{\hat{0}}{}^{0}\ e_{\hat{0}}{}^{0}\ \bar{\beta}^{g}_{00} = -\bar{\beta}^{g}_{\hat{0}\hat{0}}\ , \tag{2.20}$$

$$\begin{aligned} \bar{\beta}^{\tilde{g}}_{\hat{0}\alpha} &= \tilde{e}_{\hat{0}}{}^{0}\ \tilde{e}_{\alpha}{}^{0}\ \bar{\beta}^{\tilde{g}}_{00} + \tilde{e}_{\hat{0}}{}^{0}\ \tilde{e}_{\alpha}{}^{i}\ \bar{\beta}^{\tilde{g}}_{0i} \\ &= -\sqrt{a}\ w_j \bar{e}_{\alpha}{}^{j} \bar{\beta}^{\tilde{g}}_{00} + \sqrt{a}\ \bar{e}_{\alpha}{}^{i} \bar{\beta}^{\tilde{g}}_{0i} \\ &= \frac{1}{a^2}\sqrt{a}\ w_j \bar{e}_{\alpha}{}^{j} \bar{\beta}^{g}_{00} - \frac{1}{a^2}\sqrt{a}\ \bar{e}_{\alpha}{}^{i}\left(w_i \bar{\beta}^{g}_{00} - a\bar{\beta}^{b}_{0i}\right) \\ &= e_{\hat{0}}{}^{0}\ e_{\alpha}{}^{i}\ \bar{\beta}^{b}_{0i} = \bar{\beta}^{b}_{\hat{0}\alpha}\ . \end{aligned} \tag{2.21}$$

The entire set of consistency conditions reads:

$$\begin{aligned} \bar{\beta}^{\tilde{g}}_{\hat{0}\hat{0}} &= -\ \bar{\beta}^{g}_{\hat{0}\hat{0}}\ , \\ \bar{\beta}^{\tilde{g}}_{\hat{0}\alpha} = \bar{\beta}^{b}_{\hat{0}\alpha} \quad &, \quad \bar{\beta}^{\tilde{b}}_{\hat{0}\alpha} = \bar{\beta}^{g}_{\hat{0}\alpha}\ , \\ \bar{\beta}^{\tilde{g}}_{\alpha\beta} = \bar{\beta}^{g}_{\alpha\beta} \quad &, \quad \bar{\beta}^{\tilde{b}}_{\alpha\beta} = \bar{\beta}^{b}_{\alpha\beta}\ , \end{aligned} \tag{2.22}$$

or, in a slightly more compact form:

$$\begin{aligned} \left(\bar{\beta}^{\tilde{g}} \pm \bar{\beta}^{\tilde{b}}\right)_{\hat{0}\hat{0}} &= -\left(\bar{\beta}^{g} \pm \bar{\beta}^{b}\right)_{\hat{0}\hat{0}}\ , \\ \left(\bar{\beta}^{\tilde{g}} \pm \bar{\beta}^{\tilde{b}}\right)_{\hat{0}\alpha} &= \pm\left(\bar{\beta}^{g} \pm \bar{\beta}^{b}\right)_{\hat{0}\alpha}\ , \\ \left(\bar{\beta}^{\tilde{g}} \pm \bar{\beta}^{\tilde{b}}\right)_{\alpha\beta} &= +\left(\bar{\beta}^{g} \pm \bar{\beta}^{b}\right)_{\alpha\beta}\ . \end{aligned} \tag{2.23}$$

In either form, the $\mathbb{Z}_2$ duality covariance is now manifestly seen. Furthermore, the job of actually verifying the consistency relations, which was rather lengthy in original form [5], is also found to simplify. This happens due to the fact that the Kaluza-Klein reduction of the relevant geometric tensors "regroups" into much simpler structures. This can be seen from the expressions in the Appendix for $R_{\hat{0}\hat{0}}, R_{\hat{0}\alpha}$ and $R_{\alpha\beta}$ as opposed to R_{00}, R_{0i} and R_{ij}. The same can be also seen, in fact, for the expressions for the Riemann tensor, so that at higher orders this simplification should continue to occur.

The analogous conditions for the beta functions differ from the above by a target diffeomorphism [5], which however can also be easily stated in the tangent frame:

$$\begin{aligned} \beta^{\tilde{g}}_{\hat{0}\hat{0}} &= -\beta^{g}_{\hat{0}\hat{0}} + \alpha' \nabla_{(\hat{0}}\xi_{\hat{0})}\ , \\ \beta^{\tilde{g}}_{\hat{0}\alpha} = \beta^{b}_{\hat{0}\alpha} - \alpha' H_{\hat{0}\alpha}{}^{\gamma}\xi_{\gamma} \quad &, \quad \beta^{\tilde{b}}_{\hat{0}\alpha} = \beta^{g}_{\hat{0}\alpha} - \alpha' \nabla_{(\hat{0}}\xi_{\alpha)}\ , \\ \beta^{\tilde{g}}_{\alpha\beta} = \beta^{g}_{\alpha\beta} - \alpha' \nabla_{(\alpha}\xi_{\beta)} \quad &, \quad \beta^{\tilde{b}}_{\alpha\beta} = \beta^{b}_{\alpha\beta} - \alpha' H_{\alpha\beta}{}^{\gamma}\xi_{\gamma}\ , \end{aligned} \tag{2.24}$$

where $\xi_a = -\frac{1}{2} e_a{}^\mu \partial_\mu \ln g_{00}$, and $(ab) = ab + ba$.

Given the extremely simple form of the above consistency conditions, one might naïvely hope that this structure would not change at higher orders. However, if one views the consistency conditions as stated through the commutator $[T, R] = 0$, it then becomes apparent that it would be very unlikely that T-duality transformations would not change: in going one order higher, the operator R is modified by the next order beta functions, succinctly, $R \equiv \alpha' R_1 \to \alpha' R_1 + \alpha'^2 R_2$, in obvious notation. The condition to be demanded for consistency, at any order, should be that the R and T motions commute, so that one should in fact expect $T \equiv T_1 \to T_1 + \alpha' T_2$ as well. This, however, should not detract from the highly nontrivial statement that a T operation can be defined at all such that $[T, R] = 0$ at higher orders. In Section 4, we will investigate this for a restricted class of backgrounds.

3. Consistency Conditions and Duality Invariance

We consider in this Section to what extent duality invariance and the consistency conditions may imply each other, and general requisites to be expected of a modified T-duality transformation at two-loop order if it is to leave the background effective action invariant. For this, it is useful to note that the integrand of the background effective action,

$$L_{\text{eff}} = \sqrt{g} e^{-2\phi} \left(\bar{\beta}^\phi - \frac{1}{4} g^{\mu\nu} \bar{\beta}^g_{\mu\nu} \right) \,, \tag{3.1}$$

can be obtained by the R operation acting on the "measure" factor $V \equiv \sqrt{g} e^{-2\phi}$ [10]:

$$-\frac{1}{2} R \left(\sqrt{g} e^{-2\phi} \right) = \sqrt{g} e^{-2\phi} \left(\bar{\beta}^\phi - \frac{1}{4} g^{\mu\nu} \bar{\beta}^g_{\mu\nu} \right) \,. \tag{3.2}$$

This is fairly simple to verify by using the fact that $\sqrt{g} \equiv \sqrt{\det g} = \sqrt{\det \bar{g}} \sqrt{g_{00}}$, and is valid at higher orders and for generic backgrounds.

From this, it becomes clear that a way to achieve T-invariance of the effective action at some higher order is to require, beyond the commutation of T and R, that V be invariant under T, for then:

$$TR \left(\sqrt{g} e^{-2\phi} \right) = T \left[\sqrt{g} e^{-2\phi} \left(\bar{\beta}^\phi - \frac{1}{4} g^{\mu\nu} \bar{\beta}^g_{\mu\nu} \right) \right] = \sqrt{\tilde{g}} e^{-2\tilde{\phi}} \left(\bar{\beta}^{\tilde{\phi}} - \frac{1}{4} \tilde{g}^{\mu\nu} \bar{\beta}^{\tilde{g}}_{\mu\nu} \right) \,, \tag{3.3}$$

while

$$R\, T \left(\sqrt{g} e^{-2\phi} \right) = R \left(\sqrt{g} e^{-2\phi} \right) = \sqrt{g} e^{-2\phi} \left(\bar{\beta}^\phi - \frac{1}{4} g^{\mu\nu} \bar{\beta}^g_{\mu\nu} \right) \,. \tag{3.4}$$

Thus, $[T,R]=0$, together with the T-invariance of V implies that the background effective action is also T-invariant. Similar reasoning also shows that, conversely, T-invariance of both V and the effective action implies that the commutator $[T,R]$ also vanishes when acting on V. This, naturally, is a weaker statement than $[T,R]=0$ as an operator identity. A simple example shows this clearly: if we take T to be the usual one-loop duality transformations, and R to be the map into the $\mathcal{O}(\alpha')$ Weyl anomaly coefficients, except for $\bar{\beta}^b_{\mu\nu}$, which we take to be wrong, say, twice the correct value, then $TV=V$, $TL_{\rm eff}=L_{\rm eff}$, and $[T,R]V=0$. Yet, $[T,R]\neq 0$, as the consistency conditions are not satisfied (and it is not the case that the consistency conditions for $\bar{\beta}^b_{\mu\nu}$ "decouple" from the invariance of the effective action, because they contain $\bar{\beta}^g_{\mu\nu}$'s, which are present in the effective action).

The above does not preclude the possibility that T-invariance of the effective action may be achieved without the invariance of V; however, a more detailed examination of the specific terms involved at $\mathcal{O}(\alpha'^2)$ shows that this possibility is considerably more complicated, so that we will choose to discard it while we can (and at $\mathcal{O}(\alpha'^2)$ we can). The corrections to T that preserve the invariance of V are rather easily found: if we assume them to be

$$\begin{aligned}\ln\tilde{g}_{00} =& -\ln g_{00} + 2\alpha' Q_0 \ ,\\ \tilde{\phi} =& \phi - \frac{1}{2}\ln g_{00} + \alpha' Q_\phi\end{aligned} \tag{3.5}$$

as well as other (for now unimportant) corrections on the remaining background fields, then

$$\begin{aligned}\sqrt{g}e^{-2\phi} =& \sqrt{\det\bar{g}}\sqrt{g_{00}}e^{-2\phi} \xrightarrow{\ T\ } \sqrt{\det\bar{g}}\sqrt{\tilde{g}_{00}}e^{-2\tilde{\phi}}\\ =& \sqrt{\det\bar{g}}\sqrt{g_{00}}e^{-2\phi+\alpha'(Q_0-2Q_\phi)} \ .\end{aligned} \tag{3.6}$$

Thus, for any Q_0 and Q_ϕ satisfying $Q_0=2Q_\phi$, V will be T-invariant. Of course, one must now verify whether any such corrections exist at all so that also $[T,R]=0$ is satisfied. This will be done in the next section, for a particular class of backgrounds.

To summarize, the following statements hold:

i) $[T,R]=0$ does *not* imply $TL_{\rm eff}=L_{\rm eff}$;

ii) $TL_{\rm eff}=L_{\rm eff}$ does *not* imply $[T,R]=0$;

iii) $[T,R]=0$ *and* $TV=V$ does imply $TL_{\rm eff}=L_{\rm eff}$;

iv) $TL_{\rm eff}=L_{\rm eff}$ *and* $TV=V$ implies $[T,R]V=0$, but does *not* imply $[T,R]=0$ in general.

The requirement motivated by string theory is that the background effective action should be T-invariant. In light of the above considerations, however, we would instead elevate to a basic principle the requirement of consistency between duality and the RG flow in the sigma model, $[T,R]=0$. Then, in order to furthermore achieve duality invariance of

the background effective action in the simplest way, one should also impose the T-invariance of $V \equiv \sqrt{g}e^{-2\phi}$.

4. Covariance at Two-Loop Order

A simple glance at the two-loop sigma model beta functions is sufficient to convince one that consistency conditions at $\mathcal{O}(\alpha'^2)$ for a generic background are extremely complicated. We choose instead to work on a more restricted background, where we will nevertheless be able to illustrate in a highly nontrivial way how consistency conditions are satisfied. We take a background previously considered by Tseytlin [3]:

$$g_{\mu\nu} = \begin{pmatrix} a & 0 \\ 0 & \bar{g}_{ij} \end{pmatrix} , \tag{4.1}$$

and $b_{\mu\nu} = 0$. Two sets of corrections to duality transformations were found in [3] such that the effective action remains invariant at two-loop order. It turns out that only one of these will furthermore satisfy the consistency conditions. These corrected transformations are:

$$\begin{aligned} \ln \tilde{a} =& - \ln a + \frac{\alpha'}{2} a_i a^i \ , \quad \tilde{g}_{ij} = g_{ij} = \bar{g}_{ij} \\ \tilde{\phi} =& \phi - \frac{1}{2} \ln a + \frac{\alpha'}{8} a_i a^i \ , \end{aligned} \tag{4.2}$$

where $a_i \equiv \partial_i \ln a$, and indices $i, j, \ldots$ are raised with the inverse metric $\bar{g}^{ij} = (\bar{g}_{ij})^{-1}$ (cf. the Appendix for further details). In particular, these modified transformations satisfy the condition $Q_0 = 2Q_\phi$ stated previously, so that V remains T-invariant. Consistency conditions follow from applying R to (4.2) (and using $[T, R] = 0$ on the l.h.s.), leading to:

$$\begin{aligned} \frac{1}{\tilde{a}} \, \tilde{\bar{\beta}}_{00} =& - \frac{1}{a} \, \bar{\beta}_{00} + \alpha' \left[a^i \partial_i \left(\frac{1}{a} \, \bar{\beta}_{00} \right) - \frac{1}{2} a^i a^j \bar{\beta}_{ij} \right] \ , \\ \tilde{\bar{\beta}}_{ij} =& \bar{\beta}_{ij} \ , \\ \tilde{\bar{\beta}}^{\phi} =& \bar{\beta}^{\phi} - \frac{1}{2a} \, \bar{\beta}_{00} + \frac{\alpha'}{4} \left[a^i \partial_i \left(\frac{1}{a} \, \bar{\beta}_{00} \right) - \frac{1}{2} a^i a^j \bar{\beta}_{ij} \right] \ . \end{aligned} \tag{4.3}$$

The anomaly coefficients appearing inside the square brackets should only be taken to $\mathcal{O}(\alpha')$, as the equations are valid to $\mathcal{O}(\alpha'^2)$. It now becomes manifest that while the above conditions certainly imply the invariance of L_{eff}, the converse is not true. However, we can use the established invariance of L_{eff} under (4.2) to our advantage, insofar as it implies that once we have shown the first two consistency conditions to hold, the third one has no other option but to be satisfied. The rest of this section is devoted to proving the first two consistency conditions in (4.3).

The $\mathcal{O}(\alpha'^2)$ Weyl anomaly coefficients of the model are [7]:

$$\begin{aligned} \bar{\beta}_{\mu\nu} &= \alpha'(R_{\mu\nu} + 2\nabla_\mu\partial_\nu\phi) + \frac{\alpha'^2}{2}R_\mu{}^{\lambda\rho\sigma}R_{\nu\lambda\rho\sigma} \ , \\ \bar{\beta}^\phi &= \frac{D-25}{6} - \frac{\alpha'}{2}(\nabla^2\phi - 2\partial_\mu\phi\partial^\mu\phi) + \frac{\alpha'^2}{16}R^{\mu\lambda\rho\sigma}R_{\mu\lambda\rho\sigma} \ . \end{aligned} \tag{4.4}$$

Using the Kaluza-Klein reduction formulas found in the Appendix, the metric anomaly coefficients translate into:

$$\begin{aligned} \frac{1}{a}\ \bar{\beta}_{00} &= \frac{\alpha'}{2}\left(-q_i{}^i + 2a^i\partial_i\phi\right) + \frac{\alpha'^2}{4}q_{ij}q^{ij} \ , \\ \bar{\beta}_{ij} &= \alpha'\left(\bar{R}_{ij} - \frac{1}{2}q_{ij} + 2\bar{\nabla}_i\partial_j\phi\right) + \frac{\alpha'^2}{2}\left(\bar{R}_{ikmn}\bar{R}_j{}^{kmn} + \frac{1}{2}q_{ik}q_j{}^k\right) \ , \end{aligned} \tag{4.5}$$

where $q_{ij} = \bar{\nabla}_i a_j + \frac{1}{2}a_i a_j$, and $\bar{\nabla}_i, \bar{R}_{ij}, \bar{R}_{ijmn}$, etc. refer to tensors calculated in the reduced metric $\bar{g}_{ij}$. Incidentally, although the expressions in the Appendix show that for a generic background most formulas are considerably simplified by referring them to the tangent space, in this restricted case no great simplification is achieved with that. We therefore choose to keep the usual indices for clarity of presentation.

Corrected duality transformations take $a \to \tilde{a}$ and $\phi \to \tilde{\phi}$ as in (4.2) and, consequently,

$$\begin{aligned} a_i &\longrightarrow \tilde{a}_i = -a_i + \alpha' a^j\bar{\nabla}_i a_j \ , \\ q_{ij} &\longrightarrow \tilde{q}_{ij} = -q_{ij} + a_i a_j + \frac{\alpha'}{2}[\bar{\nabla}_i\partial_j - \frac{1}{2}a_{(i}\partial_{j)}](a^k a_k) \ , \end{aligned} \tag{4.6}$$

where $(ij) = ij + ji$, and we only need consider terms to $\mathcal{O}(\alpha')$ in the duality transformations. The dual metric anomaly coefficients are:

$$\begin{aligned} \frac{1}{\tilde{a}}\ \tilde{\bar{\beta}}_{00} &= \frac{\alpha'}{2}\left(-\tilde{q}_i{}^i + 2a^i\partial_i\tilde{\phi}\right) + \frac{\alpha'^2}{4}\tilde{q}_{ij}\tilde{q}^{ij} \ , \\ \tilde{\bar{\beta}}_{ij} &= \alpha'\left(\bar{R}_{ij} - \frac{1}{2}\tilde{q}_{ij} + 2\bar{\nabla}_i\partial_j\tilde{\phi}\right) + \frac{\alpha'^2}{2}\left(\bar{R}_{ikmn}\bar{R}_j{}^{kmn} + \frac{1}{2}\tilde{q}_{ik}\tilde{q}_j{}^k\right) \ . \end{aligned} \tag{4.7}$$

Verification of the consistency conditions in (4.3) now requires painstaking diligence, but not too much creativity. One substitutes (4.6) into (4.7), and that and (4.5) into (4.3). Although the entire procedure is rather long, the only nontrivial step involves the use of the geometrical identity $\left[\bar{\nabla}^2, \bar{\nabla}_i\right] S = \bar{R}_{ij}\bar{\nabla}^j S$, for S a scalar.

The result we finally arrive at is that the consistency conditions, (4.3), are exactly satisfied, showing that the motions T and R commute in the space of the model, as we set out to demonstrate. In [3] another set of two-loop modified duality transformations were found which, despite not leaving the measure factor V invariant, do leave the background

effective action invariant. This second set of duality transformations, call it T', is obtained from (4.2) by a target diffeomorphism, designed such as to preserve at two-loop order the one-loop relation $\ln \tilde{a} = -\ln a$. Interestingly, we find that for T', the consistency conditions are *not* satisfied: $[T', R] \neq 0$. We do not fully understand at present why the consistency conditions are not satisfied for this second set of duality transformations, and this curious fact may well be worth investigating further. Nevertheless, as it has no bearing on the result we have shown here, we will refrain from attempting to interpret it.

Essentially all we have done until now concerns the R operation as defined through the Weyl anomaly coefficients. Ultimately, however, the motivation underlying our investigation rests in the requirement of consistency of duality symmetry with true RG motions in the space of bosonic sigma models on flat worldsheets. These RG motions are generated by beta functions, and do not exactly coincide with the R operation considered previously, although they are of course intimately related. We now present the consequences of what we have found above to the beta functions of the model and its dual.

The consistency conditions (4.3) are translated into a set of consistency conditions for the beta functions by using the fact that beta functions differ from the Weyl anomaly coefficients through (2.5). We first consider the ij component in (4.3) at $\tilde{\phi} = 0$:

$$\tilde{\beta}_{ij} = \beta_{ij} + 2\alpha' \bar{\nabla}_i \partial_j \left(\frac{1}{2} \ln a - \frac{\alpha'}{8} a_i a^i \right) \equiv \beta_{ij} - 2\alpha' \bar{\nabla}_i \xi_j \ , \tag{4.8}$$

with $\xi_\mu = -1/2 \ \partial_\mu (\ln a - \alpha'/4 \ a_i a^i)$. Taking instead $\phi = 0$ in (4.3), we find the same equation to $\mathcal{O}(\alpha'^2)$, with tilde and untilde quantities interchanged. This is also equivalent to the more symmetric form:

$$\tilde{\beta}_{ij} - \alpha' \bar{\nabla}_i \tilde{\xi}_j = \beta_{ij} - \alpha' \bar{\nabla}_i \xi_j \ . \tag{4.9}$$

These expressions represent the same consistency conditions found in [5] for the beta functions, restricted to our particular background, but now valid to $\mathcal{O}(\alpha'^2)$. Something slightly different will happen with the 00 component, however. If we again take $\tilde{\phi} = 0$, now in the first equation in (4.3), we find:

$$\frac{1}{\tilde{a}} \ \tilde{\beta}_{00} = -\frac{1}{a} \left[\beta_{00} + 2\alpha' \nabla_0 \partial_0 \left(\frac{1}{2} \ln a - \frac{\alpha'}{8} a_i a^i \right) \right] + \alpha'^2 \left[a^i \partial_i \left(\frac{\bar{\beta}^{(1)}_{00}}{a} \right) - \frac{1}{2} a^i a^j \bar{\beta}^{(1)}_{ij} \right] \ , \tag{4.10}$$

where the superscript (1) denotes the one-loop quantities (at $\tilde{\phi} = 0$)

$$\begin{aligned} \bar{\beta}^{(1)}_{00} &= \bar{\beta}^{(1-\mathrm{loop})}_{00} |_{\tilde{\phi}=0} = R_{00} + \nabla_0 \partial_0 \ln a \\ &= -\frac{a}{2} \left[\bar{\nabla}_i a^i - \frac{1}{2} a_i a^i \right] \ , \\ \bar{\beta}^{(1)}_{ij} &= \bar{\beta}^{(1-\mathrm{loop})}_{ij} |_{\tilde{\phi}=0} = R_{ij} + \nabla_i \partial_j \ln a \\ &= \bar{R}_{ij} + \frac{1}{2} \bar{\nabla}_i a_j - \frac{1}{4} a_i a_j \ . \end{aligned} \tag{4.11}$$

Using formulas from the Appendix, it is possible to see that this is equivalent to:

$$\frac{1}{\tilde{a}}\,\tilde{\beta}_{00} = -\frac{1}{a}\left(\beta_{00} - 2\alpha'\nabla_0\xi_0 - 2\alpha'^2\nabla_0\zeta_0\right)\ , \tag{4.12}$$

where

$$\begin{aligned}\zeta_i &= \partial_i(\bar{\beta}^{(1)}_{00}/a) - \frac{1}{2}\bar{\beta}^{(1)}_{ij}a^j \\ &= \frac{1}{2}\left(-\bar{\nabla}_j\bar{\nabla}_i + \frac{1}{2}q_{ij}\right)a^j\ .\end{aligned} \tag{4.13}$$

Like for the ij components, this is equivalent to the same equation with tilde and untilde quantities interchanged, and it is also equivalent to the symmetric form:

$$\frac{1}{\tilde{a}}\left[\tilde{\beta}_{00} - \alpha'\tilde{\nabla}_0\tilde{\xi}_0 - \alpha'^2\tilde{\nabla}_0\tilde{\zeta}_0\right] = -\frac{1}{a}\left[\beta_{00} - \alpha'\nabla_0\xi_0 - \alpha'^2\nabla_0\zeta_0\right]\ . \tag{4.14}$$

The possibility to write the equations above in these alternative forms is of course a consequence of the fact that, even with the modifications considered in this section, the duality transformations still correspond to a $\mathbb{Z}_2$ symmetry.

Thus, while separately both the ij and 00 components satisfy one-loop consistency conditions up to a target diffeomorphism, like the one-loop beta functions do, these are different diffeomorphisms for the ij and the 00 components. This means, in particular, that the statement that scale invariant backgrounds are mapped to scale invariant backgrounds is still not entirely transparent. Nevertheless, this statement is still true, since setting the two-loop beta functions to zero, say, in the original background, leads to

$$\beta_{\mu\nu} = \alpha'\beta^{(1-\text{loop})}_{\mu\nu} + \alpha'^2\beta^{(2-\text{loop})}_{\mu\nu} = 0 \Longrightarrow \beta^{(1-\text{loop})}_{\mu\nu} = -\alpha'\beta^{(2-\text{loop})}_{\mu\nu}\ . \tag{4.15}$$

When substituted in (4.10), this shows that all components of the beta functions now satisfy the one-loop consistency conditions with the *same* diffeomorphism, up to terms of $\mathcal{O}(\alpha'^3)$. This finally implies that, to the order considered, scale invariant backgrounds are mapped to scale invariant backgrounds [5],[11].

Thus, apart from the fact that formulas become more complicated at $\mathcal{O}(\alpha'^2)$, we see that nothing has gone awry, at least for the restricted class of backgrounds presented here. Not only can a duality transformation be defined such as to maintain the invariance of the background effective action (as shown in [3]), but it can at the same time be defined such as to preserve the consistency between RG flows and duality transformations in the space of the theory. Contrary to the claim in [6], we believe the dual of a sigma model continues to be a sigma model. In particular, when one model reaches a fixed point its dual also will.

5. Conclusions

In this paper we have studied the consistency between renormalization group flows and T-duality symmetry in $d=2$ bosonic sigma models. This consistency is expressed as

the requirement that T-duality and RG flows commute, when considered as motions in the parameter space of the theory. Such a requirement was known to be satisfied at $\mathcal{O}(\alpha')$[5], where it had previously been expressed as a complicated set of relations amongst the Weyl anomaly coefficients of the theory. By referring these anomaly coefficients to the target tangent space, we have been able to considerably simplify both the expression and the verification of consistency at $\mathcal{O}(\alpha')$, showing the manifest $\mathbb{Z}_2$ duality covariance of the RG flows.

This treatment also allowed us to examine at higher orders the relation between RG/duality consistency, which is motivated from a 2d field theory point of view, and duality invariance of the string background effective action, which is motivated from a string theory point of view. Insofar as our consistency relations turn out to be stronger requirements than duality invariance of the background effective action, we have proposed that such a consistency requirement be elevated to a basic principle, to be enforced at each order in perturbation theory. From it, duality invariance of the background effective action follows once the simpler requirement is made that the measure factor $\sqrt{g}e^{-2\phi}$ also be invariant under duality.

Finally, we investigated the consistency between RG flows and T-duality explicitly at two-loop order, for a restricted class of backgrounds. The fact that the beta functions are modified in going one order higher suggests that the form T-duality transformations take should also be modified at each higher order. Borrowing from the work of Tseytlin [3], we considered one of two sets of modified duality transformations for the particular backgrounds in question, and verified that the consistency conditions are exactly satisfied also at this order. The picture that emerges at two-loop order is that, although formulas become more complicated due to the perturbative corrections they receive, the consistency between RG flows and T-duality survives these complications unscathed. RG flows continue to flow covariantly with duality, and fixed points of a model are mapped to fixed points of its dual. Although at this order the duality symmetry is still $\mathbb{Z}_2$, we have not been able to express the consistency relations in a form which expresses this symmetry manifestly.

The ultimate goal in our endeavor is to understand in more precise terms the nature of quantum corrections to T-duality transformations for generic backgrounds, and fully understand the “hierarchy” (if indeed there is one) between the requirement of duality invariance of the background effective action, and the requirement of duality covariance of the RG flows in the sigma model. Even at two-loop order this is an ambitious task. In order to progress in that direction, it seems to us the next step would naturally be to consider classes of backgrounds which are more encompassing than the one considered here, if perhaps not entirely generic at first. For instance, the introduction of antisymmetric background fields is particularly interesting, as it would bring in for the first time the

scheme dependence present in the $\mathcal{O}(\alpha'^2)$ beta functions. It would furthermore allow for a comparison with the special cases provided by WZW models, where some exact results are known (cf. [12] and related work). The particular class of backgrounds we initially have in mind contain a generic metric and no torsion in the original target, corresponding to a block diagonal metric plus torsion in the dual. We have already initiated such an investigation.

We finally note that work similar in spirit to what we have presented here has also been done in entirely different contexts, namely for lattice spin systems, and the quantum Hall effect [13]. In the string/sigma model context, for D+1-dimensional backgrounds with D isometries, the preservation of duality symmetry, in this case $O(D, D)$, at two-loop order has recently been considered in [14]. T-duality has also been studied for massive sigma models, and thus away from conformal points, in [15] and, in the case of open strings, its interplay with the RG has been considered in [16].

Appendix A. Kaluza-Klein Reduction

For the sake of the assiduous reader who would like to reproduce our results, we list below all quantities relevant for our computations. We write a generic background metric $g_{\mu\nu}$ as in (2.15), and the components of the antisymmetric background tensor $b_{\mu\nu}$ as $b_{0i} \equiv w_i$ and b_{ij}. In this notation, barred quantities refer to the metric $\bar{g}_{ij}$.

1) *Inverse metric*: $g^{00} = 1/a + v_i v^i$, $g^{0i} = -v^i$, $g^{ij} = \bar{g}^{ij}$. On decomposed tensors, indices $i, j, \ldots$ are raised and lowered with the metric $\bar{g}_{ij}$ and its inverse. With the metric decomposition (2.15) we also have $\det g = a \det \bar{g}$.

2) *Connection coefficients*:

$$\begin{aligned}
\Gamma^0_{00} &= \frac{a}{2} v^i a_i \ , \ \Gamma^0_{i0} = \frac{a}{2}\left[\frac{a_i}{a} + v^j a_j v_i + v^j F_{ji}\right] \ , \\
\Gamma^i_{00} &= -\frac{a}{2} a^i \ , \ \Gamma^i_{0j} = -\frac{a}{2}\left[F^i{}_j + a^i v_j\right] \ , \\
\Gamma^0_{ij} &= -\bar{\Gamma}^k_{ij} v_k + \frac{1}{2}(\partial_i v_j + \partial_j v_i + a_i v_j + a_j v_i) - \frac{a}{2} v^k \left[v_j F_{ik} + v_i F_{jk} - a_k v_i v_j\right] \ , \\
\Gamma^i_{jk} &= \bar{\Gamma}^i_{jk} + \frac{a}{2}\left[v_j F_k{}^i + v_k F_j{}^i - a^i v_j v_k\right] \ ,
\end{aligned} \tag{A.1}$$

where $a_i = \partial_i \ln a$, $F_{ij} = \partial_i v_j - \partial_j v_i$.

3) *Ricci tensor*:

$$\begin{aligned}
R_{00} &= -\frac{a}{2}\left[\bar{\nabla}_i a^i + \frac{1}{2} a_i a^i - \frac{a}{2} F_{ij} F^{ij}\right] \ , \\
R_{0i} &= v_i R_{00} + \frac{3a}{4} a^j F_{ij} + \frac{a}{2} \bar{\nabla}^j F_{ij} \ , \\
R_{ij} &= \bar{R}_{ij} + v_i R_{0j} + v_j R_{0i} - v_i v_j R_{00} - \frac{1}{2} \bar{\nabla}_i a_j - \frac{1}{4} a_i a_j - \frac{a}{2} F_{ik} F_j{}^k \ .
\end{aligned} \tag{A.2}$$

4) *Riemann tensor*:

$$
\begin{aligned}
R_{i0k0} &= -\frac{a}{2}\left(\frac{1}{2}a_i a_k + \bar{\nabla}_i a_k + \frac{a}{2}F_i{}^l F_{lk}\right) \ , \\
R_{ijk0} &= v_j R_{i0k0} - v_i R_{j0k0} - \frac{a}{2}\bar{\nabla}_k F_{ij} - \frac{a}{2}\left(a_k F_{ij} + \frac{1}{2}a_j F_{ik} - \frac{1}{2}a_i F_{jk}\right) \ , \\
R_{ijkm} &= \bar{R}_{ijkm} + R_{ijk0}v_m + R_{jim0}v_k + R_{mkj0}v_i + R_{kmi0}v_j \ , \\
&\quad - R_{m0j0}v_i v_k + R_{k0j0}v_i v_m - R_{k0i0}v_j v_m + R_{m0i0}v_j v_k \ , \\
&\quad - \frac{a}{4}\left(F_{im}F_{kj} + F_{ki}F_{mj} + 2F_{ji}F_{mk}\right).
\end{aligned}
\tag{A.3}
$$

5) *Torsion*:

$$
\begin{aligned}
H_{0ij} &= -\partial_i w_j + \partial_j w_i \equiv -G_{ij} \ , \\
H_{ijk} &= \partial_i b_{jk} + \partial_j b_{ki} + \partial_k b_{ij} \ ,
\end{aligned}
\tag{A.4}
$$

and all other components vanish. For the one-loop beta function the following quantities are needed:

$$
\begin{aligned}
H_{0\mu\nu}H_0{}^{\mu\nu} &= G_{ij}G^{ij} \ , \\
H_{0\mu\nu}H_i{}^{\mu\nu} &= -2G_{ij}G^{jk}v_k - H_{ijk}G^{jk} \ , \\
H_{i\mu\nu}H_j{}^{\mu\nu} &= 2\left(\frac{1}{a} + v_m v^m\right)G_i{}^k G_{jk} - 2v^k v^m G_{ik}G_{jm} + 2H_{km(i}G_{j)}{}^k v^m \\
&\quad + H_{ikm}H_j{}^{km} \ ,
\end{aligned}
\tag{A.5}
$$

and

$$
\begin{aligned}
\nabla_\mu H^\mu{}_{0i} &= \bar{\nabla}^j G_{ji} - aG_{ij}F^{jk}v_k + \frac{1}{2}G_{ij}a^j - \frac{a}{2}F^{jk}\left(H_{ijk} + v_i G_{jk}\right) \ , \\
\nabla_\mu H^\mu{}_{ij} &= \bar{\nabla}^k\left(H_{kij} + v_k G_{ij}\right) - \frac{1}{2}\left[G_i{}^k \bar{\nabla}_{(k}\ v_{j)} - G_j{}^k \bar{\nabla}_{(k}\ v_{i)}\right] - \frac{a}{2}v_{[i}H_{j]km}F^{km} \\
&\quad + v_{[i}G_{j]k}\left(a^k - aF^{km}v_m\right) + \frac{1}{2}a^k H_{kij} + \frac{1}{2}v_m a^m G_{ij} - \frac{1}{2}F_{[i}{}^k G_{j]k} \ ,
\end{aligned}
\tag{A.6}
$$

where $[ij] = ij - ji$ and $(ij) = ij + ji$.

6) *Dilaton terms*:

$$
\begin{aligned}
\nabla_0\partial_0\phi &= \frac{a}{2}a^i\partial_i\phi \ , \\
\nabla_0\partial_i\phi &= \frac{a}{2}\left(F^j{}_i + a^j v_i\right)\partial_j\phi \ , \\
\nabla_i\partial_j\phi &= \bar{\nabla}_i\partial_j\phi - \frac{a}{2}\left(v_i F_j{}^k + v_j F_i{}^k - a^k v_i v_j\right)\partial_k\phi \ .
\end{aligned}
\tag{A.7}
$$

(7) *Tangent space geometrical tensors*:

When referred to the tangent space, the Ricci tensor becomes

$$\begin{aligned} R_{\hat{0}\hat{0}} &= -\frac{1}{2}\left[\bar{\nabla}_i a^i + \frac{1}{2}a_i a^i - \frac{a}{2}F_{ij}F^{ij}\right] \ , \\ R_{\hat{0}\alpha} &= \bar{e}_\alpha^{\ i}\left[\frac{3\sqrt{a}}{4}a^j F_{ij} + \frac{\sqrt{a}}{2}\bar{\nabla}^j F_{ij}\right] \ , \\ R_{\alpha\beta} &= \bar{e}_\alpha^{\ i}\bar{e}_\beta^{\ j}\left[\bar{R}_{ij} - \frac{1}{2}\bar{\nabla}_i a_j - \frac{1}{4}a_i a_j - \frac{a}{2}F_{ik}F_j^{\ k}\right] \ , \end{aligned} \tag{A.8}$$

where $\bar{e}_\alpha^{\ i}$ is the inverse vielbein for the metric $\bar{g}_{ij}$. Likewise, the Riemann tensor is

$$\begin{aligned} R_{\alpha\hat{0}\beta\hat{0}} &= -\frac{1}{2}\bar{e}_\alpha^{\ i}\bar{e}_\beta^{\ j}\left(\frac{1}{2}a_i a_j + \bar{\nabla}_i a_j + \frac{a}{2}F_i^{\ s}F_{sj}\right) \ , \\ R_{\alpha\beta\gamma\hat{0}} &= -\frac{\sqrt{a}}{2}\bar{e}_\alpha^{\ i}\bar{e}_\beta^{\ j}\bar{e}_\gamma^{\ k}\left(a_k F_{ij} + \frac{1}{2}a_j F_{ik} - \frac{1}{2}a_i F_{jk} + \bar{\nabla}_k F_{ij}\right) \ , \\ R_{\alpha\beta\gamma\delta} &= \bar{e}_\alpha^{\ i}\bar{e}_\beta^{\ j}\bar{e}_\gamma^{\ k}\bar{e}_\delta^{\ m}\left(\bar{R}_{ijkm} - \frac{a}{4}(F_{im}F_{kj} + F_{ki}F_{mj} + 2F_{ji}F_{mk})\right) . \end{aligned} \tag{A.9}$$

References

[1] For a review and further references, see A. Giveon, M. Porrati and E. Rabinovici, *Phys. Rept.* **244** (1994) 77.

[2] T.H. Buscher, *Phys. Lett.* **194B** (1987) 59; *Phys. Lett.* **201B** (1988) 466.

[3] A.A. Tseytlin, *Mod. Phys. Lett.* **A6** (1991) 1721.

[4] J. Panvel, *Phys. Lett.* **284B** (1992) 50; J. Balog, P. Forgács, Z. Horváth and L. Palla, `hep-th/9601091`.

[5] P.E. Haagensen, *Phys. Lett.* **382B** (1996) 356.

[6] J. Balog, P. Forgács, Z. Horváth and L. Palla, *Phys. Lett.* **388B** (1996) 121.

[7] A.A. Tseytlin, *Phys. Lett.* **178B** (1986) 34; *Nucl. Phys.* **B294** (1987) 383.

[8] K.A. Meissner and G. Veneziano, *Phys. Lett.* **267B** (1991) 33; *Mod. Phys. Lett.* **A6** (1991) 3397.

[9] J. Maharana and J.H. Schwarz, *Nucl. Phys.* **B390** (1993) 3.

[10] A.A. Tseytlin, *Phys. Lett.* **194B** (1987) 63.

[11] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B274** (1986) 349.

[12] E. Kiritsis, *Nucl. Phys.* **B405** (1993) 109.

[13] P.H. Damgaard and P.E. Haagensen, `cond-mat/9609242`; C.P. Burgess and C.A. Lütken, `cond-mat/9611070`.

[14] K.A. Meissner, *Phys. Lett.* **392B** (1997) 298.

[15] S.F. Hewson and N.D. Lambert, `hep-th/9703143`.

[16] H. Dorn and H.-J. Otto, `hep-th/9702018`.